

Statistical behaviour of isotropic and anisotropic fluctuations in homogeneous turbulence

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Abstract

We review recent progresses on anomalous scaling and universality in anisotropic and homogeneous hydrodynamic turbulent flows. As a central matter, we discuss the validity and the limits of classical ideas of statistical isotropy restoration. Finally, we comment on a still open issue, the observed different scaling behaviour of longitudinal and transverse velocity increment moments in purely statistically isotropic ensemble.

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1. Introduction

Statistical restoration of symmetries of the Navier–Stokes equations is at the base of modern theories of turbulence [1]. The presence of geometrical boundaries or obstacles, or the way energy is injected in the flow usually break exact symmetries of the equation of motion. However, for high enough Reynolds number flows, those symmetry properties are supposed to be locally restored in a statistical sense, e.g. only for average quantities. Local homogeneity and local isotropy deserve a particular attention, since they are key features of theoretical approaches to turbulence and transport models. While there has been just a few attempts to make a systematic theory for deviations from statistical homogeneity [2] (see also [3,4] for recent results), it is remarkable that about isotropy restoration, there has been a considerable progress in the last years, as reviewed in Ref. [5]. As a result of this progress, effective data analysis and systematic theoretical studies have been possible, such as to separate isotropic from anisotropic features

of turbulent homogeneous statistical fluctuations. Motivation for these researches is related to puzzling experimental and numerical observations, dubbed *persistence of anisotropies*, contradicting classical expectations of recovery of isotropy [6–9]. Persistence of anisotropy accounts for the fact that purely anisotropic adimensional quantities, such as the skewness of velocity gradients transverse to the mean flow do not decay, but remain order $\mathcal{O}(1)$ at very large Reynolds numbers.

On a more general perspective, a proper understanding of scaling behaviours in statistically homogeneous but anisotropic flows is crucial to assess the *universality* of statistical properties of hydrodynamic turbulence [10].

Some crucial steps toward a clear understanding of the statistics of anisotropic fluctuations have been done in the context of Kraichnan models [11–13], simple linear models for passive transport of scalar or vector quantities by homogeneous, isotropic and Gaussian velocity fields, in the presence of large-scale homogeneous but anisotropic forcing [14–18]. While we cannot review these works, it is worth to recall their fundamental results. Isotropic and anisotropic fluctuations can be characterized by different scaling exponents, whose statistical importance is governed by their degree of anisotropy; these exponents are independent of large-scale forcing or

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boundary conditions, hence universal (see also [19] for a discussion of the case in the presence of an anisotropic and inhomogeneous forcing). The symmetry breaking and peculiar nature of the forcing is revealed in the coefficients appearing in the scaling laws, which are not universal.

In the absence of analytical approaches able to show the validity of these results in the full nonlinear problem, accurate experimental [20–24] and numerical [25–31] measurements become of fundamental importance. Encompassing all results achieved so far, or attempting an historical review of different approaches to homogeneous isotropic – where anisotropic effects are neglected, – and anisotropic turbulence, go beyond our goal and can be found in Ref. [5]. Also we mention that different approaches to anisotropy, mainly focused on large scale flow properties, have been extensively studied in Ref. [32].

Our focus is on small-scale anisotropy. At this purpose, we will first discuss the use of the $SO(3)$ decomposition of statistical observables in terms of their projections on different sectors of the group of rotations in three dimensions [33] (see also Ref. [34] for a review focusing on experimental measurements). The use of $SO(3)$ decomposition, providing a complete basis for angular decomposition, enables us to systematically describe the limits of the idea of *isotropy restoration* at sufficiently small scales (or sufficiently high Reynolds number), as postulated by the Kolmogorov theory [1]. Key working hypothesis, that we will discuss in the sequel, is that forcing has its support at scales much larger than those of the inertial range.

Secondly, we will consider the specific case of large-scale shear flows, for which a theoretical prediction for the dimensional scaling of exponents of velocity increment moments (structure functions) of any order and any degree of anisotropy can be done [35]. Results point to the existence of universal isotropic and anisotropic scaling exponents, deviating from their dimensional values. Anomalous scaling and universality of turbulent fluctuations appear as two concepts intimately related, as highlighted in Kraichnan models.

Finally, we will consider statistically homogeneous and *isotropic* turbulent flows, which can be realized with some degree of accuracy in experiments and in numerics. Compared to strongly anisotropic situations as those encountered in geophysical or plasma applications, they represent a much simpler problem. However, a large number of studies [7,22,36–44] report possible different behaviours for the longitudinal and transverse velocity structure functions in 3D flows, for moments high enough. These results contradict our expectations (for second and third moments, analytical constraints resulting from isotropy and incompressibility impose the same scaling to longitudinal and transverse fluctuations). Recent observations will be here reviewed, and commented in the light of the $SO(3)$ decomposition.

The paper is organized as follows. Section 2 recalls the theoretical framework to deal with weak anisotropic fluctuations and the notion of isotropy recovery; this is done by means of the $SO(3)$ decomposition, briefly sketched.

In Section 3, by means of the specific case of homogeneous shear flows, a dimensional argument for the scaling of anisotropic fluctuations is recalled and compared to numerical observations. Last Section 4, before concluding remarks, is devoted to the issue of longitudinal and transverse structure functions scaling in homogeneous isotropic turbulence.

2. Anisotropic hierarchy and the $SO(3)$ decomposition

The starting point of a systematic approach to small-scale anisotropic turbulence is to suppose that both boundary conditions and forcing – which break the invariance under rotation of the Navier–Stokes equations [45], – give a dominant contribution only at large scales, while the transfer of fluctuations from large to small scales is driven by the rotational invariant terms of the equations of motion. This is equivalent to say that anisotropy is only weakly affecting the statistical properties of the turbulent field under exam. Strongly sheared flows constitute a noticeable exception [46,47], as well as magneto-hydrodynamic (MHD) flows in the presence of a mean field for which we still do not have clear evidences [48,49]. However, when the previous hypothesis of large-scale forcing holds, we can study the behaviour of velocity correlation functions in the *inertial range*, at scales $\eta \ll r \ll L$ where η is the dissipation scale and L is the scale of the forcing.

To separate isotropic from anisotropic contributions, it is useful to consider their projections on the irreducible representations of the $SO(3)$ group. As a standard observable, we consider the two-points homogeneous second-order structure function

$$S^{\alpha\beta}(\mathbf{r}) \equiv \langle (v_\alpha(\mathbf{r}) - v_\alpha(0))(v_\beta(\mathbf{r}) - v_\beta(0)) \rangle.$$

The decomposition of $S^{\alpha\beta}(\mathbf{r})$ in terms of the eigenfunctions of the rotational operator is made by a set of functions labelled with the usual indices $j = 0, 1, \dots$ and $m = -j, \dots, +j$, corresponding to the total angular momentum and to the projection of the total angular momentum on an arbitrary direction, respectively.

For scalars quantities, as the longitudinal structure function, $S_L^{(2)}(\mathbf{r}) \equiv \langle [(\mathbf{v}(\mathbf{r}) - \mathbf{v}(0)) \cdot \hat{\mathbf{r}}]^2 \rangle$, the set of basis functions are the spherical harmonics, $Y_{jm}(\hat{\mathbf{r}})$. For a generic p th order tensor, in addition to indices j and m , another index q is necessary, labelling different irreducible representations within each fixed j sector [5,33]. It is easy to show that there are only $q = 1, \dots, 6$ irreducible representations of the $SO(3)$ group for the space of two-indices symmetrical tensors as $S^{\alpha\beta}(\mathbf{r})$. Accordingly, the second order structure function can be exactly decomposed as

$$S^{\alpha\beta}(\mathbf{r}) \equiv \sum_{q=1}^6 \sum_{j=0}^{\infty} \sum_{m=-j}^{+j} S_{qjm}^{(2)}(r) B_{qjm}^{\alpha\beta}(\hat{\mathbf{r}}), \quad (1)$$

where the tensors $B_{qjm}^{\alpha\beta}(\hat{\mathbf{r}})$, defined on the unit sphere, can be seen as a generalization of the spherical harmonics to the tensorial case, and the superscript 2 in the projection $S_{qjm}^{(2)}(r)$ reminds the order of the analysed correlation function.

In Ref. [33], it has been shown that, if the forcing is at large scales, by projecting the rotational invariant part of the evolution equation for $S^{\alpha\beta}(\mathbf{r})$ on the irreducible representations of the $SO(3)$ group, we obtain a set of dynamic – unclosed – equations for each projection, in each separate sector. The terms of the equations that are not coupled with the forcing, do not depend explicitly on the index m (invariance of Navier–Stokes eqs. with respect to the orientation of the z -axis) and they mix all possible q -representations, for a given j . In other words, if forcing terms are neglected, projections obey separate dynamic equations within each j sector, which corresponds to the *foliation* of the dynamic equation for any correlation in each given sector j of the rotational group [5]. This is a powerful result since, if forcing can be neglected at small scales, it allows to analyse separately the scaling behaviour of isotropic and anisotropic fluctuations in a systematic and quantitative way by studying the behaviour of the projection coefficients $S_{qjm}^{(2)}(r)$, for any degree of anisotropy j .

Moreover, in the limit of infinite Reynolds numbers, Navier–Stokes equations become scaling invariant, sector by sector. It is thus natural to expect the existence of scaling laws characterizing each sector separately, that is:

$$S_{qjm}^{(2)}(r) \sim c_{qjm}^{(2)} r^{\xi_j^{(2)}}, \quad (2)$$

where the coefficients $c_{qjm}^{(2)}$ have to be matched with large-scale boundary conditions and forcing. Decomposition similar to that of Eq. (1) can be generalized to any p -th order tensor, associated to velocity increment moments of order $p > 2$. In principle, nothing prevents the existence of more than one exponent characterizing each separate anisotropic sector, so that the power-law in Eq. (2) has to be considered the dominant term.

When we deal with numerical or experimental data, measuring behaviour of undecomposed velocity increment moments at smaller and smaller scales might not be enough to extract clean results about scaling exponents, even for very large Reynolds number flows. Indeed the presence of anisotropic fluctuations which have not yet decayed even at very small scales, can spoil scaling, thus resulting in a superposition of different power laws.

In particular, measuring scaling properties in each separate sector becomes compulsory if we mean to assess *isotropy recovery* of turbulent statistics. Such a recovery may exist only if, for any moment of given order p , the isotropic scaling exponent is always smaller than the anisotropic ones,

$$\xi^{j=0}(p) < \xi^j(p), \quad \forall j. \quad (3)$$

More generally, a whole hierarchy among the different anisotropic exponents is naturally expected, *within* any order p :

$$\xi^{j=0}(p) \leq \xi^{j=1}(p) \leq \xi^{j=2}(p) < \dots, \quad (4)$$

where the exponents $\xi^j(p)$ are supposed to be independent of the (m, q) indices.

In models for passive advection [15,17,18], it has been demonstrated that a similar hierarchy exists, and also that

scaling exponents do not show any dependence on the q, m indices. On such basis, we expect that a hierarchy like (4) might exist also in the full hydrodynamic case, and that it is robust at changing large-scale conditions.

The independence of scaling exponents from the m index is given by the arbitrariness in defining the orientation of the coordinate axis in 3D space. That from the q index, i.e. from the set of irreducible representations of the rotation group, is much less trivial and with interesting consequences. A dependence on the q index would weaken the whole *foliation* pattern, according to which rotationally invariant properties do not depend on the set of eigenfunctions (with the same rotational properties) chosen to decompose the observables. For example, admitting that projections with different q -indices have different scaling properties could possibly explain the observed different scaling between transverse and longitudinal high-order structure functions in a isotropic statistics ($j = 0$) [22,36,39].

In Ref. [15], it has been shown for the case of passive vector advection that the differential equations for the vector field covariance foliate into independent closed equations for each j sector, which mix different irreducible representations of the $SO(3)$ group, but the scaling exponents do not exhibit any dependence on the q index. We cannot prove that the very same happens for the Navier–Stokes case, although, on a physical ground, we do not see any reason why it should not be like that.

A possible explanation for the observed discrepancy in the scaling exponents of longitudinal and transverse high-order moments might rather be sought in terms of finite Reynolds effects, which prevent from having a unique clear scaling in the inertial range. In this case the differences would become smaller and smaller by going to larger and larger Reynolds numbers. In Section 4, we will come back to this point.

Experimental and numerical measurements often deal with the scaling properties of longitudinal structure functions $S_L^{(p)}(\mathbf{r}) \equiv \langle [(\mathbf{v}(\mathbf{r}) - \mathbf{v}(0)) \cdot \hat{\mathbf{r}}]^p \rangle$. As anticipated before, these are scalar objects whose decomposition onto the eigenfunctions of the $SO(3)$ group is particularly simple,

$$S_L^{(p)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^j S_{jm}^{(p)}(r) Y_{jm}(\hat{\mathbf{r}}). \quad (5)$$

In the sequel, we will consider the scaling behaviour of low order (in p and in j) projections $S_{jm}^{(p)}(r)$.

3. Dimensional prediction for anisotropic fluctuations

A phenomenological theory for dimensional expectation of the scaling exponents of structure functions is important when we try to assess the intermittent behaviour of homogeneous turbulent fluctuations, isotropic as well as anisotropic. Lumley [50] first formulated a dimensional prediction for the scaling exponent of the second order structure function in the sector $j = 2$: $\xi_d^{(j=2)}(p = 2) = 4/3$. In Ref. [35] an argument was given for the dimensional value of scaling exponents of longitudinal structure functions of any order and any degree of anisotropy, which generalizes Lumley's one.

The idea is the following. The overall effect of the large-scale energy pumping and/or boundary conditions is to produce a large-scale anisotropic driving velocity field \mathbf{U} . This is quite natural and very often encountered in geophysical or laboratory flows. The time evolution equation for the velocity field \mathbf{v} can be written as

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + U_\beta \partial_\beta v_\alpha + v_\beta \partial_\beta U_\alpha = -\partial_\alpha p + \nu \Delta v_\alpha. \quad (6)$$

The major effect of the large-scale field is the instantaneous shear $\mathcal{I}_{\alpha\beta} = \partial_\beta U_\alpha$ which acts as an anisotropic forcing term on small scales, i.e. for scales much smaller than the typical shear-injection scale, $L_S = \sqrt{\epsilon/|\mathcal{I}|^3}$.

To build up a dimensional matching for velocity fluctuations, we first consider the equation of motion for two points quantity $\langle v_\alpha(\mathbf{r})v_\beta(0) \rangle$ in the stationary regime. Inertial and shear-induced contributions can be balanced:

$$\langle v_\delta(0)v_\beta(\mathbf{r})\partial_\beta v_\alpha(\mathbf{r}) \rangle \sim \langle \mathcal{I}_{\alpha\mu}(\mathbf{r})v_\delta(0)v_\mu(\mathbf{r}) \rangle, \quad (7)$$

to obtain a dimensional estimate of the anisotropic components of the LHS in terms of the RHS shear intensity and of the isotropic part of $\langle v(\hat{\mathbf{r}})v(0) \rangle$. Similarly for three point quantities and higher order velocity correlation. Since the shear is a large-scale *slow* quantity, a safe estimate is the following:

$$\langle \mathcal{I}_{\alpha\mu}(\mathbf{r})v_\delta(0)v_\mu(\mathbf{r}) \rangle \sim D_{\alpha\mu} \langle v_\delta(0)v_\mu(\mathbf{r}) \rangle.$$

The $D_{\alpha\beta}$ tensor, associated to the combined probability to have a given shear and a given small scale velocity fluctuation, brings angular momentum only up to $j = 2$. Composition of angular momenta ($j = 2 \oplus j - 2$), then results in the following dimensional matching:

$$S_j^{(p)}(r) \sim r|D| \cdot S_{j-2}^{(p-1)}(r), \quad (8)$$

where $S_j^{(p)}(r)$ is a shorthand notation of the projection on the j -th sector of the p -th order correlation function previously introduced, neglecting further possible dependencies on q and m indices. In Eq. (8), $|D|$ denotes the typical intensity of the shear term $D_{\alpha\beta}$ in the $j = 2$ sector. For instance, the leading behaviour of the $j = 2$ anisotropic sector of the third-order correlation is: $S_{j=2}^{(3)}(r) \sim r|D|S_{j=0}^{(2)}(r) \sim r\xi_d^2(3)$.

By using a similar argument, we can obtain dimensional predictions for the $j = 2, 4$ sectors of the fourth order structure function. The procedure is easily extended to all orders, leading to the following expression:

$$\xi_d^j(p) = \frac{(p+j)}{3}. \quad (9)$$

Direct numerical simulations (DNS), at moderate Reynolds number $Re_\lambda \sim 100$, of a fully periodic, incompressible flow with a statistically homogeneous but anisotropic large-scale energy injection have been reported in Ref. [27,35]. They can be used to test the validity of the dimensional prediction (9).

In Fig. 1 isotropic and anisotropic fluctuations, which have a signal-to-noise ratio high enough to ensure stable results, are shown. Sectors with odd j s are absent due to the parity symmetry of the longitudinal structure functions. We notice a

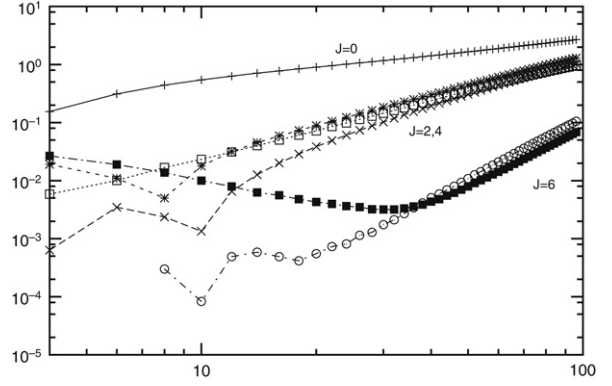


Fig. 1. Log-log plot of the second-order structure function projections $S_{jm}^{(2)}(r)$ versus the scale r , for sectors with a good signal-to-noise ratio. Sectors: $(j, m) = (0, 0)$, (+); $(j, m) = (2, 2)$, (x); $(j, m) = (4, 0)$, (empty square); $(j, m) = (4, 2)$, (*); $(j, m) = (6, 0)$, (o); $(j, m) = (6, 2)$, (black square). The statistical and numerical noise induced by the $SO(3)$ decomposition is estimated as the threshold where the $j = 6$ sector starts to deviate from the monotonic decreasing behaviour $\sim O(10^{-3})$. This figure is taken from Ref. [27]. Data come from the integration of Navier–Stokes equation for an incompressible flow, solved on a triply periodic box with 256^3 grid points; Taylor scale based Reynolds number is $Re_\lambda \sim 100$.

clear foliation in terms of the j index: sectors with the same j but different m s behave very similarly. In Table 1 the best power law fits for structure functions of orders $p = 2, 4, 6$ and sectors $j = 2, 4, 6$ are presented. It is important to notice the presence of a hierarchical organization as assumed in (4), which implies isotropy restoration at sufficiently small scales, and also that there is no saturation for the exponents as a function of the j value. Second, the measured exponents in the sectors $j = 4$ and $j = 6$ are anomalous, i.e. they differ from the dimensional estimate $\xi_d^j(p) = (j+p)/3$. This implies that isotropy is restored at small scales, but subleading anisotropic fluctuations decay slower than predicted by dimensional argument. Such difference with the dimensional scaling has been exploited in Ref. [26] to explain the puzzling results on gradients statistics mentioned in the introduction [6–9]. Persistence of anisotropy can be understood of a combined effect of anisotropy and intermittency, causing anisotropic quantities to decay at high Reynolds at much slower rates than what expected by dimensional predictions (see e.g. Ref. [51]).

Moreover, the comparison between new experimental and numerical results [24,29,30] with the data presented in Table 1 suggests that anisotropic fluctuations are indeed universal, i.e. scaling exponents for scales smaller than the typical shear length do not depend on the particular mechanism used to inject anisotropy. A different scenario may emerge if we look at scaling properties for scales larger than the typical shear length, L_S , i.e. where the external forcing mechanism cannot be neglected and therefore the foliation pattern is no longer valid [30,46,47] (consider for example turbulent convection in the Bolgiano regime). If foliation cannot be invoked, all sectors are in principle entangled and scaling properties of isotropic and anisotropic sectors may even become not universal. Further work is needed in this direction, by comparing experiments with different injection mechanisms to better highlight the statistical behaviour at scales $r \gg L_S$.

Table 1

Scaling exponents in the isotropic and anisotropic sectors obtained in Refs. [27,35] by means of DNS

p	$\xi^{j=0}(p)$	$[\xi_d^{j=0}(p)]$	$\xi^{j=2}(p)$	$[\xi_d^{j=2}(p)]$	$\xi^{j=4}(p)$	$[\xi_d^{j=4}(p)]$	$\xi^{j=6}(p)$	$[\xi_d^{j=6}(p)]$
2	0.70 ± 0.2	[0.66]	1.15 ± 0.5	[1.33]	1.65 ± 0.5	[2.00]	3.2 ± 0.2	[2.66]
4	1.28 ± 0.4	[1.33]	1.56 ± 0.5	[2.00]	2.25 ± 0.1	[2.66]	3.1 ± 0.2	[3.33]
6	1.81 ± 0.6	[2.00]	2.07 ± 0.8	[2.33]	2.60 ± 0.1	[3.33]	3.3 ± 0.2	[4.00]

Notice that values for the anisotropic sector $j = 2$, at different order moment order p , are taken from the experiments [21,22]. For the values extracted from the numerical simulation (columns $j = 0, 4, 6$), error bars are estimated on the oscillation of the local slopes. For the experimental data, error bars are given as the mismatch between the two experiments. For all sectors, the dimensional estimates for the scaling exponents $\xi_d^j(p) = (p + j)/3$ are also reported in square brackets.

4. Discussions and open issues

An issue still much debated concerns scaling in purely isotropic ensemble. Velocity tensors can be decomposed, inside the $j = 0$ isotropic sector, in q -different eigenvectors, corresponding for example to purely longitudinal, purely transverse or mixed longitudinal and transverse fluctuations [5]. Purely longitudinal structure functions are given by $S_L^{(p)}(r) \equiv \langle [\delta \mathbf{v}(\mathbf{r}) \cdot \hat{\mathbf{r}}]^p \rangle$; purely transverse structure functions are: $S_T^{(p)}(r) \equiv \langle [\delta \mathbf{v}(\mathbf{r}_T)]^p \rangle$ (where $\mathbf{r}_T \cdot \mathbf{v} = 0$). As previously discussed, arguments based on $SO(3)$ decomposition do not distinguish among scaling properties inside a given j sector. If different scaling are observed among transverse and longitudinal fluctuations within the $j = 0$ sector for statistically isotropic flows, new ideas must be presented to explain them. In Fig. 2, we show a comparison between logarithmic local slopes of order $p = 8$ and $p = 4$ in the ESS sense [52,53], of longitudinal and transverse structure functions [36,54]:

$$\zeta(p, r) = \frac{d \log S_{L,T}^{(p)}(r)}{d \log S_{L,T}^{(2)}(r)},$$

for data issuing from two different numerical simulations. This is equal to the ratio of the scaling exponent of the p -th order longitudinal (transverse) structure function to that of the second order longitudinal (transverse) one. The two DNS are *ideally* statistically isotropic since the forcing mechanism is such, and the flow has periodic boundary conditions. Residual anisotropic contribution due to the discretized nature of the numerical grid and to statistical fluctuations in the velocity statistics induced by the forcing, can be quantified and result to be very small in the data shown here. Still, in the inertial range the two datasets agree in showing a detectable difference between the longitudinal and the transverse scaling exponents.

This discrepancy is an open theoretical issue, not explainable using standard symmetry argument in homogeneous and isotropic turbulence [5]. If this is an effect due to finite-Reynolds number or a result which remains true even for most intense turbulent realizations is yet not known (see also [39] for a discussion on this point).

In recent years, many detailed observations about anisotropic turbulence have been collected. These have also given a burst for developing a systematic theory for disentangling isotropic and anisotropic fluctuations in the case of statistically homogeneous turbulent flows. We have now observation of statistical restoration of isotropy in passive transport and hydrodynamic turbulence. However, isotropy is recovered at a slower

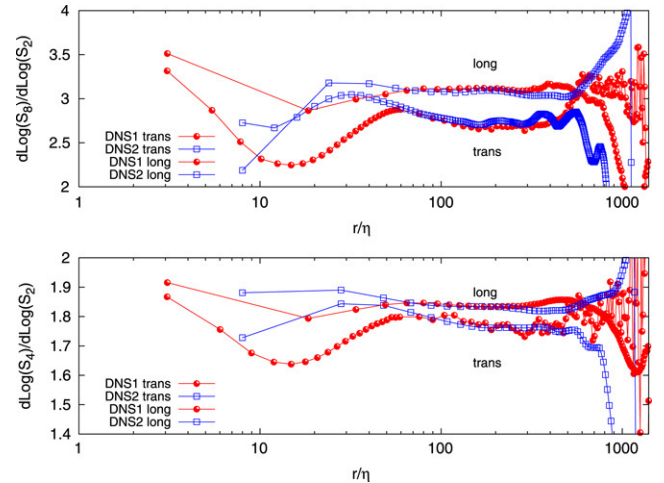


Fig. 2. Top figure: Log-log plot of the local slopes, in ESS, of the 8th-order longitudinal (top lines) and transverse (bottom lines) structure functions versus the scale r/η , as obtained from DNS data of incompressible turbulence from Ref. [36], (DNS1: circles). These are compared with DNS data obtained for a slightly compressible turbulent flow, as reported in Ref. [54], (DNS2: squares). Bottom figure: The same but for the fourth-order longitudinal and transverse structure functions. Note the good agreement of the two datasets in the inertial range, $r \gg \eta$, where they display the same mismatch between the longitudinal and transverse moments. The discrepancy between DNS1 and DNS2 data close to the dissipative scale, $r/\eta \sim 1$, is due to the fact that the two simulations have different small-scale dissipation mechanisms. DNS1 data refer to an incompressible turbulent flow, with normal viscous dissipation; numerical resolution is 1024^3 grid points and Taylor scale-based Reynolds number is $Re_\lambda \sim 460$. DNS2 data refer to a slightly compressible turbulent flow with Mach number ~ 0.3 ; numerical resolution is 1856^3 grid points, and estimated Reynolds number is $Re_\lambda \sim 600$. In this run, there are two mechanisms of kinetic energy dissipation. The most important is the transformation of kinetic energy into heat via compressible effects; the second is a numerical smoothing of steep velocity gradients tuned to filter out local numerical instabilities. The latter is important only at scales of the order of the grid spacing.

rate than expected by dimensional argument, due to intermittency. Also, there are evidences that anisotropic exponents, as well as isotropic ones, are anomalous and universal. Numerical and experimental results match with the analytical results obtained in linear model for passive advection, where it has been shown the existence of a hierarchy of exponents depending on the anisotropy degree, as well as the intermittency and universality of these exponents.

Our understanding of anisotropic turbulence is, however, based on the idea that boundary conditions and forcing contribute only at large scales, and do not break rotational invariance at scales in the inertial range. This might not be always true, particularly if we consider the case of MHD

turbulence, for which there are observations that anisotropy can grow going at smaller and smaller scales [48]. Similarly, shear flows in the production range or turbulent convection in the Bolgiano regime may possess strong departure from the sort of phenomenology observed within the *foliation* scheme.

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